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Model misspecification analysis for bond options and Markovian hedging strategies

Mireille Bossy · Rajna Gibson · Francois-Serge Lhabitant · Nathalie Pistre · Denis Talay

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Abstract In this paper, we analyse the model misspecification risk of Markovian hedging strategies for discount bond options. We show how to decompose the Profit and Loss that results from model misspecification, and emphasize the importance of the position's gamma in order to control it. We further provide mathematical results on the distribution of the forward Profit and Loss function for specific univariate term structure models. Finally, we run numerical simulations for options' hedging strategies in order to examine the sensitivity of the forward Profit and Loss function with respect to the volatility of the forward rate curve, the frequency of the position rebalancing and the characteristics of the position being hedged.

M. Bossy · D. Talay
INRIA, Sophia-Antipolis, France

R. Gibson (✉)
Swiss Banking Institute, University of Zurich, Plattenstrasse 14, Zurich 8032, Switzerland
e-mail: rgibson@isb.unizh.ch

F. S. Lhabitant
Kedge Capital, 10 Stanhope Gate, London W1K 1AL, UK

F. S. Lhabitant
University of Lausanne – School of Economics and Business Administration (HEC-Lausanne), Lausanne 1015, Switzerland

F. S. Lhabitant
EDHEC Business School, Nice Cedex, France

N. Pistre
CDC Asset Management, Paris, France

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1 Introduction

Model risk has been an ongoing source of concern for financial institutions trading or managing interest rate derivatives. Indeed, choosing between the numerous term structure modelling approaches offered by the academic literature is not trivial.[1] Should one use an equilibrium or a no-arbitrage based model, a single versus a multi-factor model? Should one specify the state variables' process as being time-homogeneous or not? Should the model be calibrated or not to the current term structure of interest rates to preserve the global consistency of the valuation process? These questions are driven both by economic as well as by regulation-based considerations to manage interest rate risk in order to prevent losses and to optimize capital requirements.

While a large number of highly reputable banks, financial institutions and hedge funds have suffered from extensive losses due to interest rate model risk, the academic literature on the topic is rather poor. Most papers are limited to a typology of model risk (see Crouhy et al. 1998; Gibson et al. 1999) or focus on the fit or the prediction of option prices (see for instance Flesaker 1993; Amin and Morton 1994; Bakshi et al. 1997; Moraleda and Vorst 1997; Buhler et al. 1999; Jagannathan et al. 2003; Longstaff et al. 2001a, b). It is only recently that a few papers have started examining the hedging of interest rate contingent claims for some specific products and models (see for instance Andersen and Andreasen 2001; Gupta and Subrahmanyam 2001; Fan et al. 2001; Driessen et al. 2003).

In this paper, we provide a simple framework in which one can characterize and decompose at any date the P&L resulting from model misspecification for an agent trading and hedging interest rate contingent claims. Model misspecification is one of the key components of model risk especially for interest rate derivatives for which a multitude of pricing and hedging models coexist. Curiously, it has received only minimal attention in the literature when compared to the wide body of research devoted to the problems raised by the proper estimation of term structure models (see Gallant and Tauchen 1997; Campbell et al. 1997).

We illustrate the applicability of our framework by focusing on discount bond option hedging strategies performed in a Markovian term structure setting. Model misspecification is analysed by assuming that the "true" term structure is characterized by a model belonging to the Markovian short-term interest rate class or the univariate Heath, Jarrow, Morton (1992)—hereafter HJM—class, while an agent or market maker chooses to rely and hedge his short option position with another model. Our objective is to quantify the agent's model risk based profit and loss (hereafter P&L) given that he uses a self-financing pseudo-replicating strategy and to analytically or numerically solve and characterize its distribution.

[1] For a survey on interest rate models, see Gibson et al. (2006).

In reality, the "true" model is unknown and thus, it is more appropriate to consider model misspecification analysis as being performed with respect to a benchmark model selected by the investor, the risk controller or the regulator. Although there is not—yet—unanimity among regulators or financial institutions with respect to a standard or benchmark interest rate model, many financial institutions rely on some existing external or internal models as benchmarks in order to vet new models. Furthermore, regulators already analyse banks' balance sheet interest rate exposure with respect to a benchmark model. This is the spirit in which this analysis of model misspecification is also conducted in order to provide a useful framework to practitioners in the field. Hong and Li (2005) take a different approach and rely on non-parametric specification tests to discriminate between alternative term structure models.

The main contribution of this paper is to provide an analytical decomposition of the model risk based P&L into three distinct terms: an initial pricing error (date 0), a current pricing error (evaluated as of the current date t), and a cumulative hedging error. This last term consists of the agent's erroneous "gamma" (calculated with the "incorrect model") multiplied by the squared deviation between the agent's and the true forward rate volatility curves specifications. This decomposition emphasizes the need to control gamma exposures in order to minimize model misspecification. Such a monitoring is required in order to minimize model risk without inducing volatility gaming strategies with respect to the benchmark model. In addition, we provide some analytical properties of the model risk forward P&L function for some simple forward rate volatility specifications, and use numerical simulations to show that its magnitude is economically significant. Finally, those simulations also emphasize the dramatic impact of discrete portfolio rebalancing on the magnitude of the model risk P&L moments. The proposed framework is fairly general since it applies to discount bond option hedging strategies performed in any univariate Markovian term structure setting.

The structure of the paper is the following. In Sect. 2, we provide a general derivation of the decomposition of the model risk P&L function,[2] which is then applied in Sects. 3 and 4 to the hedging of discount bond options within Markovian short-term interest rates models and within the HJM model, respectively. Sections 5 and 6 rely on numerical simulations to illustrate the evolution of the moments and quantiles of the model risk P&L function in the case of specific term structure models. Section 7 discusses possible extensions of the proposed model risk assessment framework and Sect. 8 concludes the presentation.

2 A general description of the model risk P&L function for Markovian hedging strategies

In the sequel, we consider a continuous-time economy. We take as given a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a Brownian motion W and its filtration $\mathcal{F} = \{\mathcal{F}_t; t \in [0, T^*]\}$ where $T^* > 0$ is a finite time horizon. $\mathbb{P}$ corresponds to the

[2] We would like to thank an anonymous referee for suggesting to include this general decomposition of the model risk P&L.

historical probability measure. The set $\mathcal{F}_t$ represents the whole information available at time t. We make the usual assumption that $\mathcal{F}_0$ is trivial and $\mathcal{F}_{T^*} = \mathcal{F}$.

Consider a primary asset with price process S and a saving account with price process F defining a no-arbitrage and complete market. To the saving account corresponds a change of numeraire. For the sake of consistency with the notation used below for bond options, we denote by the price S_t^F of the primary asset expressed in this numeraire. We suppose that, up to a change in probability from $\mathbb{P}$ to a new probability $\mathbb{P}^F$, the price S_t^F is a $\mathcal{F}$-martingale and defines a complete and perfect market in the Harrison and Kreps (1979) and Harrison and Pliska (1981) sense. For example, within the Black and Scholes paradigm, S^F is the discounted stock price which is a martingale under the risk neutral probability. We emphasize that the change of numeraire and the use of $\mathbb{P}^F$ are not absolutely necessary in the sequel, but they dramatically simplify calculations.[3]

Consider an option on the primary asset with maturity T^O and payoff function ϕ. At all time $0 \le t \le T^O$, the perfectly hedging portfolio consists of H_t^0 units of the saving account and H_t units of the primary asset:

$$V_t = H_t^0 F_t + H_t S_t.$$

In the numeraire F this can be rewritten as

$$V_t^F = H_t^0 + H_t S_t^F.$$

As this means that we hold H_t^0 units of an asset with the constant value 1, the self-financing condition implies

$$V_t^F = V_0^F + \int_0^t H_\theta d S_\theta^F. \tag{1}$$

Here we suppose that the process (H_t) is non-anticipating and admissible in the sense that the preceding stochastic integral is well defined.

Since the martingale S^F is supposed to define a complete market and thus satisfy the martingale representation property, equality (1) provides a characterization of the process H. However, without further constraints on the model, H can only be implicitly defined, or represented, by means of quantities such as conditional expectations of Malliavin derivatives as in the Clark–Ocone formula, which are not easily tractable from a numerical point of view. Thus, the agent needs to use a model that allows him to obtain solutions for the strategies aimed to hedge the option at least approximately.

Usually, traders consider a Markovian environment, that is, they construct their strategies by means of an observed one dimensional[4] process (ρ_t) which they decide to model as a Markov process satisfying $F_t = \bar{h}(t, \rho_t)$ and $S_t^F = \bar{g}(t, \rho_t)$, for some functions $\bar{g}$ and $\bar{h}$. Thus, to design the hedging strategy, the trader considers a market

[3] We also emphasize that checking for market completeness in the context of term structure models may lead to delicate questions, as can be seen from the last subsection in the Appendix.

[4] Here we suppose that ρ_t is a one dimensional process for the sake of notational simplicity.

governed by a probability $\overline{\mathbb{P}}$ and a Brownian motion $\bar{W}$ and its Brownian filtration $\bar{\mathcal{F}}$, and (ρ_t) is supposed to solve an equation of the type

$$d\rho_t = \bar{\beta}(t, \rho_t)dt + \bar{\gamma}(t, \rho_t)d\bar{W}_t, \tag{2}$$

for some smooth enough functions $\bar{\beta}, \bar{\gamma}$. We emphasize that $\bar{W}$ is introduced to justify the formula that the trader uses to define the hedging strategy. This formula relies on a PDE which reflects the Markov structure of the model, and thus involves the coefficients $\bar{\beta}$ and $\bar{\gamma}$ of the dynamics (2) but not the Brownian motion $\bar{W}$.

Models which are commonly used satisfy the following constraint: there exists a probability $\overline{\mathbb{P}}^F$ equivalent to $\overline{\mathbb{P}}$ under which S^F is a martingale satisfying the martingale representation property and solving

$$dS_t^F = \frac{\partial \bar{g}}{\partial x}(t, \rho_t)\bar{\gamma}(t, \rho_t)d\bar{W}_t^F \tag{3}$$

for some $\bar{\mathbb{P}}^F$-Brownian motion $\bar{W}^F$ and functions $\bar{g}$ and $\bar{\gamma}$. Notice that, under appropriate conditions on all the functions involved in the model,

$$d\rho_t = \tilde{\bar{\beta}}(t, \rho_t)dt + \bar{\gamma}(t, \rho_t)d\bar{W}_t^F, \tag{4}$$

where the new drift coefficient $\tilde{\bar{\beta}}$ can be expressed in terms of $\bar{\beta}$, $\bar{\gamma}$ and $\bar{h}$.

In order to obtain a representation for (H_t) suitable for numerical approximations, we seek a smooth function $\bar{\pi}(t, x)$ solution to the partial differential equation

$$\frac{\partial \bar{\pi}}{\partial t}(t, x) + \bar{\mathcal{L}}_t^{\rho} \bar{\pi}(t, x) = 0 \tag{5}$$

with boundary condition (remember that $V_{T^O}^F = \frac{1}{F_{T^O}}\phi(F_{T^O}\ S_{T^O}^F))$

$$\bar{\pi}(T^O, x) = \frac{\phi(\bar{h}(T^O, x)\bar{g}(T^O, x))}{\bar{h}(T^O, x)},$$

where $\bar{\mathcal{L}}_t^{\rho}$ is the infinitesimal operator of the Markov process solution to (4).

Within the classical Black and Scholes paradigm we have $\rho_t = S_t^F = e^{-rt}S_t$ and $\bar{\pi}(t, x) = e^{-rt}v(t, e^{rt}x)$, where $v(t, x)$ is the solution of the standard Black and Scholes PDE. More generally, if one chooses to use a model with constant interest rate and stock price with a volatility of the type $\bar{\sigma}(t, S_t)$ then ρ is chosen as S^F itself.

In order to deduce $\bar{H}_t$ we emphasize a subtle feature that the model needs to include. As shown in Revuz and Yor (2005), a tractable sufficient condition to ensure that the process S^F solution to (3) defines a complete market is that

$$\forall 0 \leq t \leq T^O,\ \frac{\partial \bar{g}}{\partial x}(t, \rho_t)\bar{\gamma}(t, \rho_t) > 0\ a.s.$$

Under this constraint on the model that the trader chooses to use, one has

$$\bar{\pi}(T^O, \rho_{T^O}) = \bar{\pi}(0, \rho_0) + \int_0^{T^O} \frac{\partial \bar{\pi}}{\partial x}(\theta, \rho_\theta)\bar{\gamma}(\theta, \rho_\theta) d\bar{W}_\theta^F,$$

and, in view of (3), risk can be eliminated through a delta-hedge given by

$$\bar{H}_t = \frac{\partial \bar{\pi}}{\partial x}(t, \rho_t) \left\{ \frac{\partial \bar{g}}{\partial x}(t, \rho_t) \right\}^{-1}.$$

Then the value (expressed in the numeraire and in the true market) of the self-financed "pseudo replicating" strategy of the agent satisfies

$$d\overline{V}_t^F = \overline{H}_t dS_t^F. \tag{6}$$

Now define the model risk P&L function as

$$P\&L_t^F = \overline{V}_t^F - V_t^F. \tag{7}$$

In order to estimate that function, we suppose that, in the true world, the process (ρ_t) is a (not necessarily Markov) Itô process whose dynamics under $\mathbb{P}^F$ are given by

$$d\rho_t = \beta_t dt + \gamma_t dW_t^F$$

for some adapted processes β and γ. Set

$$\mathcal{L}_t^\rho \bar{\pi}(t, \rho_t) := \beta_t \frac{\partial \overline{\pi}}{\partial x}(t, \rho_t) + \frac{1}{2}(\gamma_t)^2 \frac{\partial^2 \overline{\pi}}{\partial x^2}(t, \rho_t). \tag{8}$$

Applying Itô's lemma to $\overline{\pi}(t, \rho_t)$ and using (5) and (6), one obtains:

$$\begin{aligned} d\overline{\pi}(t, \rho_t) &= \frac{\partial \overline{\pi}}{\partial t}(t, \rho_t)dt + \mathcal{L}^\rho \overline{\pi}(t, \rho_t)dt + \frac{\partial \overline{\pi}}{\partial x}(t, \rho_t)\gamma_t dW_t^F \\ &= (\mathcal{L}_t^\rho - \overline{\mathcal{L}}_t^\rho)\bar{\pi}(t, \rho_t)dt + d\overline{V}_t^F, \end{aligned}$$

so that[5]

$$\overline{V}_t^F - \overline{V}_0^F = \overline{\pi}(t, \rho_t) - \overline{\pi}(0, \rho_0) + \int_0^t (\overline{\mathcal{L}}_\theta^\rho - \mathcal{L}_\theta^\rho)\overline{\pi}(\theta, \rho_\theta) d\theta.$$

[5] Notice that the next equality induces that $\bar{H}_t^O = \overline{V}_t^F - \bar{H}_t$ can be expressed without a stochastic integral which makes the computations more tractable.

Thus we have

$$P\&L_t^F = \overline{V}_t^F - V_t^F = \overline{V}_0^F - \overline{\pi}(0, \rho_0) + \overline{\pi}(t, \rho_t) - V_t^F + \int_0^t (\overline{\mathcal{L}}_\theta - \mathcal{L}_\theta)\overline{\pi}(\theta, \rho_\theta) d\theta. \tag{9}$$

At maturity T^O, this equality simplifies to

$$P\&L_{T^O}^F = \overline{V}_{T^O}^F - V_{T^O}^F = \overline{V}_0^F - \overline{\pi}(0, \rho_0) + \int_0^{T^O} (\overline{\mathcal{L}}_\theta^\rho - \mathcal{L}_\theta^\rho)\overline{\pi}(\theta, \rho_\theta) d\theta. \tag{10}$$

Notice that, if (ρ_t) is a Markov process, that is, if $\beta_t = \beta(t, \rho_t)$ and $\gamma_t = \gamma(t, \rho_t)$ for some functions β and γ, then $\mathcal{L}_t^\rho$ is the classical infinitesimal generator of (ρ_t) and $V_t^F = \pi(t, \rho_t)/F_t$ where $\pi(t, x)$ solves a PDE similar to (5) with $\mathcal{L}_t^\rho$ replacing $\bar{\mathcal{L}}_t^\rho$. In particular, if the model error only affects the volatility term in S_t^F, the integral in (9) will reduce to an expression involving the position's gamma.

We now emphasize that equality (9) holds true under $\mathbb{P}^F$ almost surely, and therefore almost surely under the historical probability as well.

We also emphasize that the above P&L decomposition offers an insightful economic interpretation. Indeed, we observe that the P&L at time $t < T^O$ consists of three distinct terms:

- the first term represents the initial pricing error which the agent incurs at inception of his strategy. This term drops out if the agent can sell the option at the price given by his own model, or if the agent calibrates his model to match observed market prices.
- the second term represents the model pricing error at any given date t chosen to compute the forward P&L. At the maturity of the option, this term vanishes, since the terminal payoff of the contingent claim is model-independent.
- the last term represents the cumulative impact of the model error on the hedging strategy up to date t.

The above results are fairly general since they apply to any Markov specification, provided the above mentioned smoothness hypotheses on the parabolic PDEs hold. In light of the significant volume of trading in interest rate derivatives and due to the importance of interest rate risk management in financial institutions, we have chosen to focus in the following on the analysis of model risk in the case of discount bond written options. In addition, the analytical structure of formula (9) is richer in the case of discount bond written options than in the case of standard stock options under the Black and Scholes paradigm. Indeed, to price interest rate contingent claims, one would like to use families of general models to which the 'true model' may belong but, for the sake of efficiency, one often prefers to use a simplified model which is easy to calibrate and leads to simple hedging strategies. We now describe in detail this particular setting and analyse model risk for two different families of univariate Markovian term structures models: models based on Markovian short-term interest rates and the HJM models.

3 Hedging discount bond options in univariate short-term rate diffusion models

3.1 A brief review of bond pricing in short-term rate diffusion models

We now consider a trader who uses the following framework to build his hedging strategies: the market supports a short-term interest rate (r_t) which is an adapted process such that the savings account process (B_t) satisfies

$$B_t = \exp\left(\int_0^t r_\theta d\theta\right) \tag{11}$$

for all $0 \leq t \leq T^*$. Then (see, e.g., chap. 12 in Musiela and Rutkowski 1997), for all maturity $T \leq T^*$, the arbitrage-free family $(B(t, T))$ of bond prices relative to (r_t) is defined by

$$B(t, T) = \bar{\mathbb{E}}\left(\exp\left(-\int_t^T r_\theta d\theta\right) \mid \bar{\mathcal{F}}_t\right). \tag{12}$$

Here, the trader chooses a framework where the local expectations hypothesis holds, so that the probability $\bar{\mathbb{P}}$ is a martingale measure (the reader may easily adapt the following to frameworks with risk premia). Then, there exists an adapted process $(\bar{\sigma}^*(t, T))$ such that[6]

$$dB(t, T) = B(t, T)(r_t dt - \bar{\sigma}^*(t, T)d\bar{W}_t). \tag{13}$$

It is important to observe that, in general, the process $(\bar{\sigma}^*(t, T), 0 \leq t \leq T)$ is provided by the Brownian martingale representation theorem (see, e.g., Sect. 3.4 in Karatzas and Shreve 1991), and thus, in general, cannot be explicited. In addition, when it is possible to explicit it, the representation is generally not so simple since $\bar{\sigma}^*(t, T) = \bar{\psi}(t, B(t, T))$ for some deterministic function $\bar{\psi}$, and thus, in this general framework, the process $(B(t, T), 0 \leq t \leq T^*)$ is not considered as a Markov process. Therefore, mathematical analysis and numerical simulations cannot easily be performed. Fortunately, an acceptable assumption on the process (r_t) simplifies the setting: namely, we now specify that (r_t) is under $\overline{\mathbb{P}}$ the unique strong solution of a stochastic differential equation of the type

$$r_t = r_0 + \int_0^t \bar{\beta}(s, r_s)ds + \int_0^t \bar{\gamma}(s, r_s)d\bar{W}_s \tag{14}$$

for some smooth enough functions $\bar{\beta}$ and $\bar{\gamma}$. We also suppose that there exists a smooth solution $\bar{u}_T(t, x)$ to the following Partial Differential Equation driven by the infinitesimal generator $\bar{\mathcal{L}}_t^r$ of (r_t):

$$\begin{cases} \dfrac{\partial \bar{u}_T}{\partial t}(t, r) + \bar{\mathcal{L}}_t^r \bar{u}_T(t, r) - r\, \bar{u}_T(t, r) = 0, \quad t < T, \;\; r \in \mathbb{R}, \\ \bar{u}_T(T, r) = 1, \quad r \in \mathbb{R}. \end{cases} \tag{15}$$

[6] See Proposition 12.2.1 in Musiela and Rutkowski (1997).

Applying Itô's formula to $\bar{u}_T(\alpha, r_\alpha)\exp(-\int_t^\alpha r_s ds)$ and using (15), under suitable conditions on $\partial\bar{u}_T/\partial r$ and integrability conditions on the process $(\int_t^\alpha r_s ds,\ t \leq \alpha \leq T)$, one obtains that

$$\left(\bar{u}_T(\alpha, r_\alpha)\exp\left(-\int_t^\alpha r_s ds\right) - \bar{u}_T(t, r_t),\ t \leq \alpha \leq T\right)$$

is a martingale, from which, choosing $\alpha = T$ and using (12), one has

$$B(t, T) = \bar{u}_T(t, r_t). \tag{16}$$

We thus necessarily have that $\bar{\sigma}^*(t, T)$ defined in (13) satisfies

$$\bar{\sigma}^*(t, T) = -\frac{1}{\bar{u}_T(t, r_t)}\frac{\partial \bar{u}_T}{\partial r}(t, r_t)\bar{\gamma}(t, r_t). \tag{17}$$

In view of $\bar{u}_T(T, r) \equiv 1$, one has $\partial\bar{u}_T/\partial x(t, r)$ tends to 0 when t goes to T. We thus obtain the classical pull-to-par property.

In order to simplify future calculations, for $T^O \leq T$, we introduce the T^O-forward price $B^F(t, T)$ of the bond of maturity T. This price is defined by a change of numeraire:

$$B^F(t, T) := \frac{B(t, T)}{B(t, T^O)}. \tag{18}$$

An easy calculation shows that

$$dB^F(t, T) = (\bar{\sigma}^*(t, T^O) - \bar{\sigma}^*(t, T))B^F(t, T)d\bar{W}_t^F, \tag{19}$$

where the process

$$\bar{W}_t^F := \bar{W}_t - \int_0^t \frac{1}{\bar{u}_T(s, r_s)}\frac{\partial \bar{u}_T}{\partial r}(s, r_s)ds \tag{20}$$

is a Brownian motion for some Forward risk neutral probability measure $\overline{\mathbb{P}}^F$.

3.2 Option hedging under model risk and model risk P&L

We next consider a European option written on the zero coupon bond $B(t, T)$, with maturity $T^O < T$, exercise price K, and payoff at maturity of the type $\phi(B(T^O, T))$, where ϕ is a given function. We suppose that the market consists of the zero coupon bonds $B(t, T^O)$ and $B(t, T)$, the first one being the numeraire.

Notice that (19) can be rewritten as

$$dB^F(t, T) = \left(\frac{1}{\bar{u}_T(t, r_t)}\frac{\partial \bar{u}_T}{\partial r}(t, r_t) - \frac{1}{\bar{u}_{T^O}(t, r_t)}\frac{\partial \bar{u}_{T^O}}{\partial r}(t, r_t)\right)\bar{\gamma}(t, r_t)\bar{u}_T(t, r_t)d\bar{W}_t^F, \tag{21}$$

and therefore, to preserve the framework developed in Sect. 2 (see, in particular, Eq. 3), it suffices to choose $\rho_t \equiv r_t$ and to check that $B^F(t,T)$ satisfies the martingale representation property.[7]

In order to price the option and implement the hedging strategy, the trader uses

$$\begin{cases} \frac{\partial \bar{\pi}}{\partial t}(t,r) + \bar{\mathcal{L}}_t^r \bar{\pi}(t,r) = 0, & t < T^O, \ \ r \in \mathbb{R}, \\ \bar{\pi}(T,r) = \phi \circ u_T(T_0, r), & r \in \mathbb{R}, \end{cases} \tag{22}$$

where $\bar{\mathcal{L}}_t^r$ stands for the infinitesimal generator of (r_t) under $\overline{\mathbb{P}}^F$, that is, in view of (14) and (20),

$$\bar{\mathcal{L}}_t^\rho \bar{\pi}(t,r) := \left(\bar{\beta}(s,r) + \frac{1}{\bar{u}_T(s,r)} \frac{\partial \bar{u}_T}{\partial r}(s,r)\right) \frac{\partial \bar{\pi}}{\partial r}(t,r) + \frac{1}{2} \frac{\partial^2 \bar{u}_T}{\partial r^2}(t,r).$$

Then one can apply (9).

If the true market is governed by such a short-term rate model with coefficients β and γ, then one can define $u_T(t,r)$, $\pi(t,r)$ and $\mathcal{L}_t^\rho$ in an obvious way and use the observations following Eq. 10 to define the decomposition of the model risk P&L function.

4 Hedging of discount bond options in a univariate HJM framework

4.1 A brief review of the HJM framework

In this section we suppose that the trader, in order to define his hedging strategy, chooses to use the term structure dynamics characterized by a univariate Markov model belonging to the HJM class of term structure models. For the trader, at time t, the instantaneous forward interest rate for riskless and instantaneous borrowing or lending at date $T \geq t$ satisfies the following equality under $\overline{\mathbb{P}}$:

$$f(t,T) = f(0,T) + \int_0^t \bar{\mu}_f(s,T)ds + \int_0^t \bar{\sigma}_f(s,T)d\bar{W}_s. \tag{23}$$

HJM show that in the absence of arbitrage, the drift of the forward rates under the equivalent martingale measure, with the money-market account as numeraire, is completely determined by the volatility:[8]

$$\bar{\mu}_f(s,T) = \bar{\sigma}_f(s,T)\bar{\sigma}_f^*(s,T), \quad \text{with } \bar{\sigma}_f^*(s,T) := \int_s^T \bar{\sigma}_f(s,u)du. \tag{24}$$

[7] For the latter point, please see the Appendix.

[8] Note that given the key role played by the volatility specification in the HJM framework, $\bar{\sigma}_f(t,T)$ jointly accounts for estimation risk and misspecification risk.

In the following, we limit the mathematical treatment of the problem to Markovian univariate HJM models[9] and specify $\bar{\sigma}_f(\cdot, T)$ as a positive deterministic function of t and T. In such a model the price at any time $0 \leq t \leq T$ of a zero coupon bond maturing at date T with a face value of one currency unit is given by

$$B(t, T) = \exp\left(-\int_t^T \bar{f}(t, s)ds\right). \tag{25}$$

Using Itô's lemma and the standard no-arbitrage arguments, one can show that the partial differential equation governing the bond price is

$$\begin{cases} dB(t, T) = r_t B(t, T)dt - \bar{\sigma}_f^*(t, T)B(t, T)d\bar{W}_t, \\ B(T, T) \ = 1, \end{cases} \tag{26}$$

where r_t denotes the instantaneous risk-free rate. We define the T^O-forward price $B^F(t, T)$ as in Sect. 3. Standard calculations show that $B^F(t, T)$ is the solution of a linear stochastic differential equation similar to (26).

We thus define the Forward risk neutral probability measure $\overline{\mathbb{P}}^F$ through the Girsanov transformation removing the drift in the stochastic differential equation satisfied by $B^F(t, T)$. Under $\overline{\mathbb{P}}^F$ we have

$$dB^F(t, T) = (\bar{\sigma}_f^*(t, T^O) - \bar{\sigma}_f^*(t, T))B^F(t, T)d\bar{W}_t^F, \tag{27}$$

for some $\overline{\mathbb{P}}^F$-Brownian motion $(\bar{W}_t^F)$. Notice that $B^F(\cdot, T)$ is a martingale and that, as we have supposed that $\bar{\sigma}_f(t, u)$ is positive, it obviously satisfies the martingale representation theorem. Thus, in the present framework, $B^F(\cdot, T)$ defines a complete market.

4.2 Option hedging under model risk and model risk P&L

As before, we consider a European option on the zero coupon bond $B(t, T)$, with maturity $T^O < T$, exercise price K, and payoff at maturity of the type $\phi(B(T^O, T))$. We suppose that the market consists of the zero coupon bonds $B(t, T^O)$ and $B(t, T)$, the first one being the numeraire. Observe that the payoff at maturity is equal to $\phi(B^F(T^O, T))$, so that, in view of (27), the T^O-forward prices allow us to work within the framework developed in Sect. 2. We may choose $\rho_t \equiv B^F(t, T)$ and thus $\bar{g}(x) \equiv x$.

The agent's replicating portfolio contains $\bar{H}_t^O$ units of the discount bond of maturity T^O and $\bar{H}_t$ units of the discount bond of maturity T, for a total value of

$$\bar{V}_t := \bar{H}_t^O B(t, T^O) + \bar{H}_t B(t, T),$$

[9] See Jeffrey (1995) for a review of the general conditions on the volatility specifications in HJM models under which the Markovian property is retained.

or equivalently

$$\bar{V}_t^F = \bar{H}_t^O + \bar{H}_t B^F(t,T).$$

As noticed above, here $\bar{g}(x) \equiv x$, so that the quantity $\bar{H}_t$ is given by the delta of the contingent claim according to the univariate Markov HJM model with the volatility specification $\bar{\sigma}_f(t,T)$:

$$\bar{H}_t = \frac{\partial \bar{\pi}_{\bar{\sigma}_f}}{\partial x}\Big(t, B^F(t,T)\Big) \tag{28}$$

where, in view of (27), the function $\bar{\pi}_{\bar{\sigma}_f}$ solves the parabolic partial differential equation (5) with

$$\bar{\mathcal{L}}_t^{\rho}\bar{\pi}(t,x) := \frac{1}{2}x^2(\bar{\sigma}_f^*(t,T^O) - \bar{\sigma}_f^*(t,T))^2 \frac{\partial^2 \bar{\pi}_{\bar{\sigma}_f}}{\partial x^2}(t,x). \tag{29}$$

Hence, given that in the real market $B^F(t,T)$ is an Itô process of the type (27), the P&L function satisfies Eq. 9. Now, if the real market is governed by an HJM model parameterized by functions $\sigma_f^*(t,T)$, then (9) becomes

$$\begin{aligned} P\&L_t^F &= \bar{V}_0^F - \bar{\pi}_{\bar{\sigma}_f}(0, B^F(0,T)) + \bar{\pi}_{\bar{\sigma}_f}(t, B^F(t,T)) - \pi_{\sigma_f}(t, B^F(t,T)) \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2 \bar{\pi}_{\bar{\sigma}_f}}{\partial x^2}(\theta, B^F(\theta,T)) B^F(\theta,T)^2 \\ &\quad \times \Big\{(\bar{\sigma}_f^*(\theta,T) - \bar{\sigma}_f^*(\theta,T^O))^2 - (\sigma_f^*(\theta,T) - \sigma_f^*(\theta,T^O))^2\Big\}\, d\theta. \end{aligned} \tag{30}$$

We now show that under some conditions on the convexity or concavity of the option payoff, the model risk P&L can be signed explicitly. Consider the hedging of a single option with a smooth payoff in the absence of initial pricing errors. Notice that

$$\bar{\pi}_{\bar{\sigma}_f}(t,x) = \overline{\mathbb{E}}^F \phi(\bar{B}_{t,x}^F(T^O,T)), \tag{31}$$

where the expectation is computed under $\overline{\mathbb{P}}^F$ and

$$\begin{cases} d\bar{B}_{t,x}^F(\theta,T) = (\bar{\sigma}_f^*(\theta,T^O) - \bar{\sigma}_f^*(\theta,T))\bar{B}_{t,x}^F(\theta,T)\,d\bar{W}_\theta^F, \quad t \le \theta \le T^O, \\ \bar{B}_{t,x}^F(t,T) = x. \end{cases} \tag{32}$$

Since (32) is a linear equation we have

$$\begin{aligned} \frac{\partial^2 \bar{\pi}_{\bar{\sigma}_f}}{\partial x^2}(t,x) &= \overline{\mathbb{E}}^F \Bigg[\frac{d^2\phi}{dx^2}(\bar{B}_{t,x}^F(T^O,T)) \\ &\quad \times \exp\Bigg(2\int_t^{T^O}\int_{T^O}^T \bar{\sigma}_f(s,u)du\, d\bar{W}_s^F - \int_t^{T^O}\Big(\int_{T^O}^T \bar{\sigma}_f(s,u)du\Big)^2 ds\Bigg)\Bigg]. \end{aligned}$$

Since ϕ is of class $\mathcal{C}^2(\mathbb{R})$, the forward P&L at time T^O can thus be re-expressed as follows:

$$
\begin{aligned}
P\&L^F_{T^O} &= \frac{1}{2}\int_0^{T^O} \bar{\mathbb{E}}^F\left[\frac{d^2\phi}{dx^2}(\bar{B}^F_{t,x}(T^O,T))\right.\\
&\quad\left.\left.\times\exp\!\left(2\int_t^{T^O}\!\!\int_{T^O}^T \bar{\sigma}_f(s,u)du\, d\bar{W}^F_s - \int_t^{T^O}\left(\int_{T^O}^T \bar{\sigma}_f(s,u)du\right)^2 ds\right)\right]\right|_{t=\theta,x=B^F(\theta,T)}\\
&\quad\times\left\{\left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2 - \left(\int_{T^O}^T \sigma_f(\theta,u)du\right)^2\right\} B^F(\theta,T)^2\, d\theta.
\end{aligned}
\tag{33}
$$

This follows directly from the definition of σ_f^* in Eq. 24 and from the forward P&L definition in Eq. 30.

Proposition 1 *If the hedged option has a strictly convex payoff, the forward model risk P&L at date T^O can be signed explicitly. Its sign depends on the difference between the integral of the benchmark forward rate volatility and the integral of the agent's volatility.*

Indeed, in the absence of initial pricing errors, at time $t = T^O$, Eq. 33 reduces to

$$
\begin{aligned}
P\L^F_{T^O} &= \frac{1}{2}\int_0^{T^O} \frac{\partial^2 \bar{\pi}_{\bar{\sigma}_f}}{\partial x^2}(\theta, B^F(\theta,T))B^F(\theta,T)^2\\
&\quad\times\left\{\left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2 - \left(\int_{T^O}^T \sigma_f(\theta,u)du\right)^2\right\} d\theta.
\end{aligned}
\tag{34}
$$

Thus, the sign of the forward P&L depends upon both the over/under estimation of the agent's volatility with respect to the benchmark model and the gamma of the position. In particular, if the agent adopts a conservative volatility specification, i.e.

$$
\left(\int_{T^O}^T \sigma_f(\theta,u)du\right)^2 \le \left(\int_{T^O}^T \bar{\sigma}_f(\theta,u)du\right)^2. \tag{35}
$$

his P&L is positive (negative) for a convex (concave) payoff. This strengthens the idea that in order to limit the model risk of a trader, a sensible strategy is once again to set limits on the gamma of his position.

4.3 Illustration for specific HJM term structure models

As an illustration, we now consider the case of some specific univariate HJM term structure models. The aim is to show that for some properties of the contingent claims' payoff and for some specifications of the functions σ_f^* and $\bar{\sigma}_f^*$, one can compute the P&L function or some of its moments analytically. We will examine the model risk P&L at maturity, which, if we assume no initial pricing error, consists only of the cumulative hedging error (see Eq. 30).

4.3.1 The P&L specification for a Ho and Lee strategy in a general HJM environment

We assume that the agent uses the continuous-time version of the Ho and Lee (1986) model. That is, for some strictly positive constant $\bar{\sigma}$,

$$\bar{\sigma}_f(t,T) = \bar{\sigma}, \quad \bar{\sigma}_f^*(t,T) = \bar{\sigma}(T-t), \quad \bar{\sigma}_f^*(t,T) - \bar{\sigma}_f^*(t,T^O) = \bar{\sigma}(T-T^O).$$

Proposition 2 *When the agent hedges a short position in a European call option of strike K and maturity T^O, his P&L at date T^O is bounded.*

$$\left| P\&L^F_{T^O} \right| \leq \frac{CKT^O\sqrt{T^O}}{2\sqrt{2\pi}\bar{\sigma}(T-T^O)}. \tag{36}$$

where C is defined as

$$C = \sup_{0\leq\theta\leq T^O} \left| \bar{\sigma}^2(T-T^O)^2 - \left(\int_{T^O}^{T} \sigma_f(\theta,s)ds \right)^2 \right|. \tag{37}$$

Proof See Appendix. □

This result can easily be generalized to the hedging of a short position in a European put option with the same exercise price and time to maturity, since both contingent claims have the same gamma. Obviously, in the case of a long position in a European call or put option, the agent's loss in absolute value also becomes bounded by expression (36).

4.3.2 The P&L specification for the Ho and Lee strategy in a Vasicek environment

We now suppose that the agent uses the continuous-time version of the Ho and Lee model while the benchmark term structure is governed by the Vasicek model. These two models are chosen since they represent well-known specific cases of the HJM family of univariate models, which can be distinguished by their volatility specification. For the Ho and Lee model, we have

$$\bar{\sigma}_f(t,T) = \sigma_r$$

while for the Vasicek model, we have:

$$\sigma_f(t,T) = \sigma_r \exp\left(-\kappa(T-t)\right)$$

where σ_r denotes the constant volatility of the spot rate.[10]

[10] In the following analysis of model risk, we suppose that there is no estimation risk (i.e. the estimated value of σ_r is the same for $\bar{\sigma}_f(t,T)$ and $\sigma_f(t,T)$).

Proposition 3 *An agent using a Ho and Lee model to hedge a short position in a Vasicek environment will have a positive P&L at date T^O, for all levels of σ_r and κ in all states of the world.*

Proof See Appendix. □

By symmetry, notice that hedging a long position in an option leads to a negative value of $P\&L^F_{T^O}$ for all levels of σ_r and κ in all states of the world.

4.3.3 The P&L specification for the Vasicek strategy in the Cox, Ingersoll, Ross environment

We now suppose that the agent uses the continuous-time version of the Vasicek model while the benchmark term structure is governed by the Cox et al.(1985a, b)—hereafter CIR model. Thus, suppose that the true dynamics of the short-term rate r_t satisfy the square root process

$$dr_t = \kappa^*(R_\infty - r_t)dt + \sigma_{\text{CIR}}\sqrt{r_t}dW_t.$$

suggested by Cox et al. In particular, $\gamma(t, r_t)$ in (14) is now $\sigma_{\text{CIR}}\sqrt{r_t}$. We consider the case of hedging a single short put option and compute the model risk $P\&L$ at maturity T^O. In this case, for all $0 \le t \le T$, one has

$$\frac{1}{u_T(t, r_t)}\frac{\partial u_T}{\partial r}(t, x) = -\psi(T - t)$$

with

$$\psi(\theta) = \frac{2(\exp(\gamma^*\theta) - 1)}{\gamma^* - \kappa^* + \exp(\gamma^*\theta)(\gamma^* + \kappa^*)}$$

and $\gamma^* = \sqrt{(\kappa^*)^2 + 2\sigma^2_{\text{CIR}}}$. We deduce that

$$\begin{aligned} P\&L_{T^O} = & \frac{1}{2}\int_0^{T^0} \frac{\partial^2\bar{\pi}^{\text{Vas.}}}{\partial x^2}(\theta, B^F(\theta, T))\left(B^F(\theta, T)\right)^2 \\ & \times\left\{\left(\int_{T^0}^{T} \bar{\sigma}_f^{\text{Vas.}}(\theta, u)du\right)^2 - \left(\sigma^{*,\text{CIR}}(\theta, T) - \sigma^{*,\text{CIR}}(\theta, T^0)\right)^2\right\} d\theta \end{aligned} \quad (38)$$

with

$$\bar{\sigma}_f^{\text{Vas.}}(\theta, u) = \sigma_{\text{Vas.}}\exp(-\kappa(u - \theta)),$$

and

$$\sigma^{*,\text{CIR}}(\theta, T) = -\sigma^2_{\text{CIR}} r_t \psi(T - \theta).$$

It is obviously difficult to estimate the model risk P&L as expressed in Eq. 38 mathematically. In particular, the sign of the $P\&L$ is not tractable anymore unlike in the Ho and Lee versus Vasicek situation described earlier. In Sect. 6, we will therefore rely on numerical procedures to obtain estimates of the model risk P&L moments for this specific setting.

5 Numerical illustration for the Ho and Lee strategy in a Vasicek environment

In this section, we conduct numerical simulations in the specific case examined in Sect. 4.3, where the trader uses a Ho and Lee term structure model while the benchmark term structure is governed by the Vasicek model. The aim is to quantify the magnitude of model risk on the forward P&L, and to analyze its moments and quantiles as well as the sensitivity of these values with respect to the type of position chosen by the agent under different term structure shapes. Finally, in the spirit of Figlewski and Green (1999), we also examine the impact of discrete portfolio rebalancing strategies on the magnitude of model risk. Note that for illustrative purposes, we consider the P&L at time T^O, but the analysis can easily be applied at any given date $t \leqslant T^O$, and to any univariate Markov HJM model's pairwise comparison.

5.1 Computation of the P&L at the maturity of the option

We use Monte Carlo simulations to compute statistics of the P&L, such as its expectation, its variance and its quantiles. Using Monte Carlo rather than a sample historical path allows us to generate as many data points as desired, and therefore to depart from historical patterns.

The forward P&L obtained in (34) is a random variable.[11] We are interested in computing a sample of N different realizations at date T^O corresponding to N states of the world. In each state, we simulate a trajectory of the forward price $B^F(t, T)$ between time 0 and T^O and we use it to compute the right hand side of (41). Each simulation i, for $i = 1, \ldots, N$ gives a realization $P\&L_{T^O}(i)$ of the forward P&L.

For the simulations, we have chosen the following set of parameters: the maturity of the options is respectively equal to 1 month, 6 months, and 1 year. The exercise prices are set respectively at $K = 90\%$, 100% and 110% of the initial value of the zero coupon bond. The maturity of the zero coupon bond is 5 years. The initial term structure of interest rates is either flat at a level of 7.5%, or ascending (from 5.5% to 8.3%) or descending (from 8.3% to 5.5%). The parameter κ is set to 0.04 in all simulations. The number of simulations is set to $N = 20,000$.

We have chosen to emphasize the impact of the level of σ_r, the volatility of the short-term interest rate, on the characteristics of the $P\&L^F_{T^O}$, since this is the parameter that determines model misspecification in the HJM framework. The volatility parameter varies between 0.01 (for which $P\&L^F_{T^O} \simeq 0$) and 0.12.

5.2 Simulation results for a continuously rebalanced position

The results presented in Table 1 and in Figs. 1 and 2 are obtained for a short position on a European put option written on a zero coupon bond of nominal 100. Unless otherwise stated, all the model risk $P\&L^F_{T^O}$ statistics will be expressed relative to the

[11] We will abstract from the rather delicate task of having to assume a given functional form for the risk premium. Thus, for simplicity, we set the risk premium equal to zero. Such a specification is consistent with an economy in which the local expectations hypothesis is fulfilled—see for instance Cox et al. (1981).

Table 1 Impact of the option maturity on the $P\&L^F_{T0}$ statistics in the case of a 6 month short at-the-money put option written on a 5 year zero-coupon bond

	Volatility of short term rate (σ_r)					
	$\sigma_r = 0.05$			$\sigma_r = 0.10$		
Maturity of the option (T^0)	1*M*	6*M*	1*Y*	1*M*	6*M*	1*Y*
$P\&L^F_{T0}$ Expected value	0.19	0.42	0.51	0.37	0.84	1.06
$P\&L^F_{T0}$ Volatility	0.07	0.14	0.17	0.13	0.28	0.33
$P\&L^F_{T0}$ Quantile 1%	0.06	0.13	0.16	0.12	0.27	0.36
$P\&L^F_{T0}$ Quantile 99%	0.31	0.69	0.83	0.63	1.37	1.68

The table shows the impact of the option maturity on the $P\&L^F_{T0}$ statistics. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money put on a 5 year zero-coupon bond. The option maturity can be 1 month, 6 months or 1 year. The short-term rate volatility is set to 0.05 or 0.10. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15 year and above). All results are expressed as a percentage of the initial bond price

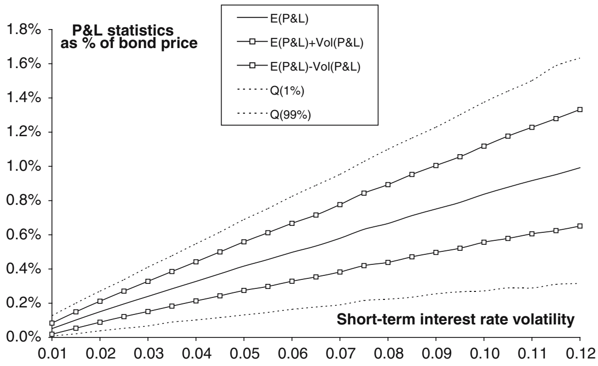

Fig. 1 Statistics for $P\&L^F_{T0}$ in the case of a 6 month short ATM put option on a 5 year zero-coupon bond (as percentage of the initial bond price). The figure shows the impact of the short-term interest rate volatility on the $P\&L^F_{T0}$ statistics. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6 month put on a 5 year zero-coupon bond. The short term rate volatility varies between 0.01 and 0.12. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15 year and above). The $P\&L^F_T$ statistics are expressed as a percentage of the initial underlying bond price

bond market price in order to maintain a uniform benchmark for the analysis of cash and derivatives positions in terms of the value of the underlying security.

Figure 1 shows that the interest rate volatility level plays a crucial role on the various statistics of the $P\&L^F_{TO}$ distribution. For the base case example of a 6-month at-the-money put option written on a five-year discount bond, we observe that $\bar{\mathbb{E}}(P\&L^F_{TO})$ is systematically positive, which illustrates the main result of Proposition 1. We notice furthermore that the $P\&L^F_{TO}$ increases almost linearly in σ_r for a short option position, and that its volatility is also increasing in the short-term rate's volatility. For the maximum level of volatility displayed ($\sigma_r = 0.12$), $\bar{\mathbb{E}}(P\&L^F_{TO})$ represents almost 1%

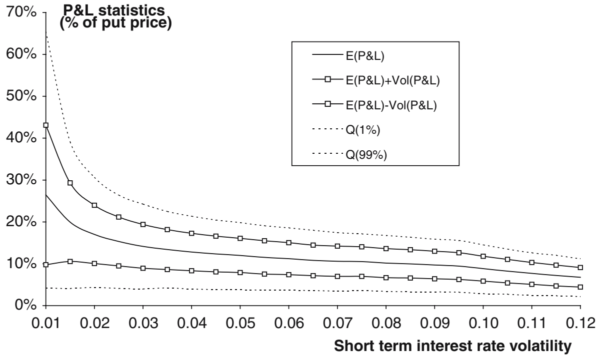

Fig. 2 Statistics for $P\&L^F_{T0}$ in the case of a 6 month short ATM put option on a 5 year zero-coupon bond (as percentage of the initial put price). The figure shows the impact of the short-term interest rate volatility on the $P\&L^F_{T0}$ statistics. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6 month put on a 5 year zero-coupon bond. The short-term rate volatility varies between 0.01 and 0.12. The term structure of interest rates is upward sloping from 5.5% (short term) to 8.3% (15 year and above). The $P\&L^F_{T0}$ statistics are expressed as a percentage of the initial put price

of the bond's nominal value. This value may appear negligible. One should however be aware of the fact that the value and sensitivity of all $P\&L^F_{TO}$ statistics are modified when expressed as a percentage of the underlying put or call market values (see Fig. 2). For instance, in the case of the put option examined in Fig. 1, the expected $P\&L^F_{TO}$ expressed as a percentage of the put market price is decreasing with respect to the short-term rate's volatility. For the minimum level of the volatility ($\sigma_r = 0.01$) displayed, $\bar{\mathbb{E}}(P\&L^F_{TO})$ represents almost 27% of the put market value, which is far from being negligible. This suggests that when analysing model risk for institutions in terms of their derivatives positions only, the importance of the model risk $P\&L^F_{T0}$ becomes highly significant because of the implied leverage.

Likewise, the 99% VaR of a single option position is highly sensitive to the level of the interest rate's volatility, ranging from 0.15% (for $\sigma_r = 0.01$) to 1.5% of the underlying position's market value (for $\sigma_r = 0.12$). Obviously, the sensitivity of the $P\&L^F_{TO}$ key statistics increases as the maturity of the option lengthens. The positive concavity of model risk with respect to the option's time to maturity increases can easily be inferred from Table 1.[12]

5.3 Simulations results for discretely rebalanced position

In practice, most institutions hedge their positions at discrete time intervals, for example at the end of the day. With respect to the continuous hedging problem, discrete rebalancing introduces a discretization error whose additional influence should be quantified. More precisely, let us consider the case of a discrete rebalancing at regular

[12] Additional simulation results illustrating the impact of the moneyness of the short put option, the shape of the term structure on the model risk $P\&L^F_{TO}$, or the case of option portfolios (spreads and straddles) are available from the authors upon request.

time intervals of size h and use the notation of Sect. 2. With discrete rebalancing, the hedging gap becomes

$$\left[\bar{V}_0 + \int_0^t \bar{H}_t dS_t^F - V_t\right] + \left[\int_0^t \bar{H}_{\tau(t)h} dS_t^F - \int_0^t \bar{H}_t dS_t^F\right],$$

where $\tau(t)$ is the integer part of t/h. In the above expression, the first term represents the model risk $P\&L_t^F$ under study in the previous sections. The second term represents the rebalancing risk profit and loss. Under the forward risk neutral probability, this term represents the difference between two martingales and its expectation is nil[13] irrespective of the value of h.

In this section, we simulate a discrete time hedging strategy: the term structure still evolves in continuous-time according to the Vasicek model, but the agent acts at discrete times with a self-financing strategy dictated by his perception of the model of the term structure, that is, a Ho and Lee model in this example. The terminal forward P&L of the agent will not correspond to the continuous forward $P\&L^F$ given by Eq. 34. In most cases, the P&L cannot be computed analytically. However, if the interval of discretization is small, we verify numerically that the discrete P&L converges towards the continuously rebalanced strategy's P&L.

In Fig. 3, we see that the expected P&L for one reallocation a day is very similar to the corresponding figure in the almost continuous trading case (as proxied by 100 reallocations a day). However, the volatility of the P&L has been exacerbated and

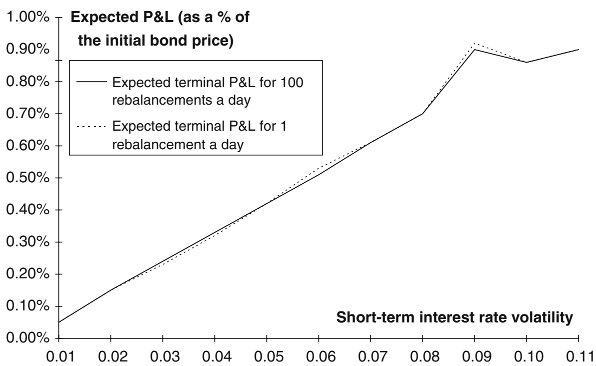

Fig. 3 Expected $P\&L_{T0}^F$ in the case of discrete rebalancing. The figure shows the impact of the short-term rate volatility on the expected $P\&L_{T0}^F$ in the case of a discrete trading strategy. We consider two possible reallocation frequencies: 100 times a day, or once a day. The $P\&L_{T0}^F$ corresponds to the results from the dynamic hedging of a short at-the-money 6 month put on a 5 year zero-coupon bond. The short-term rate volatility varies between 0.01 and 0.11. The term structure of interest rates is flat at 7.5%. The expected $P\&L_{T0}^F$ is expressed as a percentage of the initial bond price

[13] For a study of the discrete time hedging errors, see e.g. Gobet and Temam (2004) and the references therein.

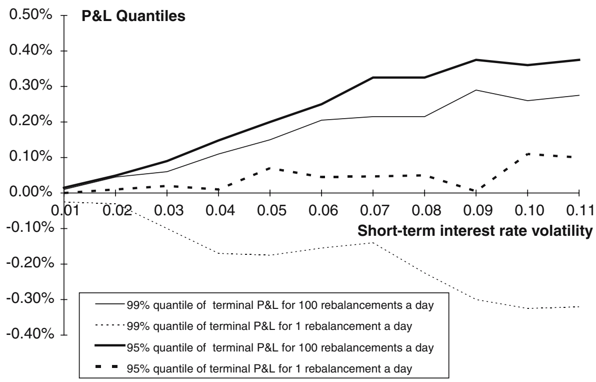

Fig. 4 95% and 99% quantiles of the $P\&L^F_{T0}$ in the case of discrete rebalancing. The figure shows the impact of the short-term rate volatility on the $P\&L^F_{T0}$ 95% and 99% quantiles in the case of a discrete trading strategy. We consider twopossible reallocation frequencies: 100 times a day, or once a day. The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6 month put on a 5 year zero-coupon bond. The short-term rate volatility varies between 0.01 and 0.11. The term structure of interest rates is flat at 7.5%. The quantiles are expressed as a percentage of the initial bond price

can increase by a factor of more than ten. The quantiles can even become negative, as illustrated in Fig. 4. This suggests that unlike in the continuous case, due to the discrete rebalancing, the model risk P&L does not remain a strictly positive (or negative, in the case of a long position) random variable. These results, like those presented by Figlewski and Green (1999), suggest that the discreteness in portfolio reallocations magnifies model risk, even for rebalancing time intervals which are commonly used by practitioners.

6 Numerical illustration for a Vasicek strategy in a CIR short-term rate environment

In this section we conduct numerical simulations in the specific case where the agent uses a Vasicek hedging strategy in a CIR term structure environment. The analytical expression of the corresponding $P\&L^F$ is given in (38). Here, we simulate the discrete time hedging strategy obtained for 100 reallocations a day.[14] As in Sect. 5.1, we use $N = 20,000$ Monte Carlo simulations in order to compute the P&L statistics plotted in Figs. 5 and 6. We choose the parameters of the strategy and of the environment to match as closely as possible those from the Ho and Lee strategy in the Vasicek environment case, in order to compare the two situations. More precisely, the parameter κ^* is set to 0.04, and the volatility of the models, $\sigma_{\mathrm{Vas}} = \sigma_{\mathrm{CIR}}\sqrt{r_0}$ varies between 0.01 and 0.12. The initial term structure of interest rates is ascending from 5.5% to 8.3%. In Figs. 5 and 6, one can observe that, for small values of the volatility, the expectation

[14] For the simulation of the CIR model for the short-term interest rate, we use a specific version of the Euler scheme proposed in Bossy and Diop (2006) for non-Lipschitz volatility model like the CIR process. The reader can also refer to Alfonsi (2005) for other simulation schemes of the CIR model.

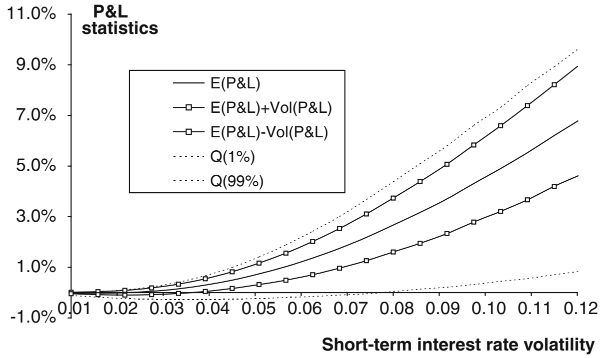

Fig. 5 Statistics of the $P\&L^F_{T0}$ when using a Vasicek-based hedging strategy in a CIR short-term rate environment. The figure shows the impact of the short-term rate volatility on the $P\&L^F_{T0}$ in the case of a discrete trading strategy (100 reallocations a day). The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6 month put on a 5 year zero-coupon bond when using a Vasicek model while the short-term rate follows a CIR process. The short-term rate volatility varies between 0.01 and 0.12. The term structure of interest rates is ascending from 5.5% to 8.3%. All statistics are expressed as a percentage of the initial bond price

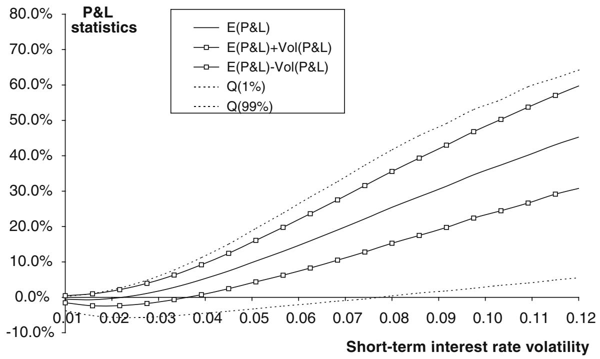

Fig. 6 Statistics of the $P\&L^F_{T0}$ when using a Vasicek-based hedging strategy in a CIR short-term rate environment. The figure shows the impact of the short-term rate volatility on the $P\&L^F_{T0}$ in the case of a discrete trading strategy (100 reallocations a day). The $P\&L^F_{T0}$ corresponds to the results from the dynamic hedging of a short at-the-money 6 month put on a 5 year zero-coupon bond when using a Vasicek model while the short-term rate follows a CIR process. The short-term rate volatility varies between 0.01 and 0.12. The term structure of interest rates is ascending from 5.5% to 8.3%. All statistics are expressed as a percentage of the initial put price

of $P\&L^F$ is negative. This is the first significant difference with the Ho and Lee vs. Vasicek situation described in Figs. 1 and 2. Moreover, all the statistics of the $P\&L^F$ are now increasing with the volatility of the short-term rate, even when the P&L is expressed as a percentage of the initial put market price. For the maximum level of volatility displayed $\sigma_{\text{Vas}} = 0.12$, the expected $P\&L^F$ represent almost 45% of the option initial value versus 27% in the Vasicek vs. Ho and Lee framework described earlier.

7 Extensions to more general models and other hedging instruments

Aside from providing a conceptual framework for decomposing the $P\&L$ related to model misspecification for interest rate sensitive claims, the approach introduced in this paper can also be applied to a fairly large class of univariate Markov term structure hedging models. We have restricted the model risk assessment framework to Markovian univariate HJM and short-term rate diffusion models. In practice, removing either assumption could result in several difficulties.

First, the estimation of stochastic volatility processes from historical interest rate data is at the least very difficult. Even deterministic time non-homogeneous volatility functions can lead to a very unrealistic pattern for the future volatility of the spot rate or can lead to overfitting to option prices, as evidenced by De Jong et al. (2001). But a more important issue for model risk assessment is the number of factors that should be used to accurately price and hedge interest rate contingent claims. Our analysis was cast in a univariate setting, which implies that the agent could hedge the option by trading only two bonds. However, there is an important and still ongoing debate in the financial literature about the adequacy of one-factor models for pricing and/or hedging interest rate contingent claims, particularly when considering instruments that are more complex than bond options and caps. For instance, Longstaff et al. (2001a) argue that the use of one-factor models for swaptions leads to significant losses because their prices directly depend on the correlation between interest rates of different maturities, while one-factor models imply perfect correlation between interest rates of different maturities. Collin-Dufresne and Goldstein (2002) evidence the presence of "unspanned stochastic volatility", i.e. there are factors driving cap and swaption implied volatilities that do not drive the term structure so that bonds do not span the fixed income market. However, these findings differ from those of Fan et al. (2003), who show that swaptions and even swaption straddles can be well hedged with LIBOR bonds alone, indicating that the swaption market is well integrated with the LIBOR-swap market. Andersen and Andreasen (2001) also find that Bermudan swaption prices change only moderately (and in fact typically decrease slightly) when the number of factors in the underlying interest rate model is increased from one to two. Their results provide support for the standard Wall-Street practice of using continuously re-calibrated one-factor models to price Bermudan swaptions, as long as calibration procedures are sufficiently comprehensive. We leave for future research to assess if the hedges set up based on this popular market approach are competitive with hedges set up using some higher order multi-factor models.

Finally, another important issue is the choice of the hedging strategy and of its instruments. In our analysis, we have hedged the option position using bonds whose maturities correspond to the cash flow dates of the option. This corresponds to the bucket hedging strategy as it is performed in practice for large books of derivative instruments. However, given the convex or concave nature of the hedged position, we could also have hedged the position using another option. This decision is likely to reduce the gamma exposure of the (incorrectly) hedged position. In a sense, our hedging illustration provides an upper bound to model misspecification in a univariate Markov framework. Alternatively, in the case of a multiple factor model, one may use factor hedging (i.e. use as many hedging instruments as factors). As illustrated by Driessen et al. (2003), the choice of the number of hedging instruments and the maturities of these hedging instruments is sometimes more important than the particular model choice. A deeper understanding of this issue in the context of model risk management constitutes an interesting extension left for further research.

8 Conclusion

In this study, we analyse the impact of one important dimension of model risk, namely model misspecification associated with the hedging of discount bond option positions for Markovian univariate term structure models. We have seen that the P&L due to model misspecification has a fairly intuitive economic interpretation and that it essentially depends on the magnitude of the position's gamma and on the squared difference between the forward rate volatility curves in the benchmark and in the trader or institution's environment. Numerical simulations were provided to highlight the fact that model misspecification is economically significant and that it is highly sensitive to the current level of interest rate volatility, to the type of positions held by the trader (simple vs. combined, long or short, in, at, or out-of-the-money), and that it also increases with the time to maturity of the position being hedged. The simulations under daily portfolio rebalancing have further shown that discrete delta hedging strategies amplify model risk. Given that the true model of the term structure is unknown, the results suggest that, in order to avoid volatility gaming strategies, the independent risk control function should set and carefully monitor limits on the derivatives position's gamma since this minimizes the model risk exposure of a financial institution.

There are several ways in which the above study could be extended. First, it would be interesting to examine the consequences on the P&L of a situation in which the trader misjudges the number as well as the specification of the factors driving the evolution of the term structure. Secondly, we ignored market disruptions and their impact on the resulting discontinuous evolution of the term structure. In particular, for emerging markets or countries in which the central bank's interventions play an important role, the jump-diffusion component of the short-term interest rate dynamics and its misspecification should also be considered. Third, issues such as estimation risk, computational risk and discretization errors also belong to the sound model risk management of interest rate derivatives' books.

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Appendix

Proof of Proposition 2 The agent's P&L at date T^O can be expressed as

$$P\&L^F_{T^O} = \frac{K}{2\sqrt{2\bar{\pi}}\bar{\sigma}(T-T^O)}\int_0^{T^O}\frac{1}{\sqrt{T^O-\theta}}$$
$$\times\exp\left(-\frac{(\log(K)-\log B^F(\theta,T)+\frac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-\theta))^2}{2\bar{\sigma}^2(T-T^O)^2(T^O-\theta)}\right)$$
$$\times\left(\bar{\sigma}^2(T-T^O)^2-\left(\int_{T^O}^{T}\sigma_f(\theta,s)ds\right)^2\right)d\theta. \tag{39}$$

Notice that

$$|P\&L^F_{T^O}| \le \frac{K}{2\sqrt{2\bar{\pi}}\bar{\sigma}(T-T^O)}\int_0^{T^O}\frac{1}{\sqrt{T^O-\theta}}$$
$$\times\left|\bar{\sigma}^2(T-T^O)^2-\left(\int_{T^O}^{T}\sigma_f(\theta,s)ds\right)^2\right|d\theta. \tag{40}$$

This is the best possible estimate, since the support of the law of the process $(B^F(\theta,T))$ is such that

$$\exp\left(-\frac{(\log(K)-\log B^F(\theta,T)+\frac{1}{2}\bar{\sigma}^2(T-T^O)^2(T^O-\theta))^2}{2\bar{\sigma}^2(T-T^O)^2(T^O-\theta)}\right)$$

can take a value which is as close as desired to 1, providing the path of $(B^F(\theta,T))$ is chosen accordingly. □

Proof of Proposition 3 We first substitute the special form of $\sigma_f(t,T)$ into expression (39), and the following expression for the model risk P&L at maturity date T^O for the seller of a European call (or a put) on a zero coupon bond follows:

$$P\&L^F_{T^O} = \frac{K\sigma_r}{2\sqrt{2\pi}(T-T^O)}\int_0^{T^O}\frac{1}{\sqrt{T^O-\theta}}$$
$$\times\left((T-T^O)^2-\frac{1}{\kappa^2}\left(e^{-\kappa(T-\theta)}-e^{-\kappa(T^O-\theta)}\right)^2\right)$$
$$\times\exp\left(-\frac{(\log(K)-\log B^F(\theta,T)+\frac{1}{2}\sigma_r^2(T-T^O)^2(T^O-\theta))^2}{2\sigma_r^2(T-T^O)^2(T^O-\theta)}\right)d\theta, \tag{41}$$

where the forward price of the zero coupon bond satisfies:

$$\frac{dB^F(t,T)}{B^F(t,T)} = \frac{\sigma_r}{\kappa}\left(e^{-\kappa(T-t)}-e^{-\kappa(T^O-t)}\right)\left[\frac{\sigma_r}{\kappa}\left(1-e^{-\kappa(T^O-t)}\right)dt+dW_t\right].$$

It can easily be shown that, for an agent who has initially sold the option, the $P\&L_{T^O}$ is a positive random variable for any level of σ_r and κ. For that purpose, let us introduce the function f defined by

$$f(\theta) = \kappa^2(T - T^O)^2 - \left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)^2$$

This function is positive. Indeed,

$$\begin{aligned} f(\theta) = &\left[\kappa(T - T^O) - \left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)\right] \\ &\times \left[\kappa(T - T^O) + \left(e^{(-\kappa(T-\theta))} - e^{-\kappa(T^O-\theta)}\right)\right]. \end{aligned}$$

Since $T > T^O$, we simply have to study the sign of

$$\left[\kappa(T - T^O) - \left(e^{-\kappa(T-\theta)} - e^{-\kappa(T^O-\theta)}\right)\right]$$

or equivalently, the sign of $p(x) - p(y)$, for $x = \kappa(T - \theta)$ and $y = \kappa(T^O - \theta)$ where $p(z) = z - e^{-z}$. Since $p(\cdot)$ is an increasing function and $x > y$, we conclude that the quantity in brackets is positive. □

On the martingale representation property of $B^F(t, T)$ for diffusion short term models

As shown in Björk (1997) and Harrison and Pliska (1981), completeness is equivalent to the so-called predictable representation property (or martingale representation property) of a certain continuous local martingale. Here, we consider the process

$$B^F(t, T) := \frac{B(t, T)}{B(t, T^O)}$$

which is a local martingale under the Forward risk neutral probability measure $\overline{\mathbb{P}}^F$ defined through the Girsanov transformation removing the drift in

$$\begin{aligned} dB^F(t, T) = &B^F(t, T)\bar{\sigma}^*(t, T^O)(\bar{\sigma}^*(t, T^O) - \bar{\sigma}^*(t, T))dt \\ &+B^F(t, T)(\bar{\sigma}^*(t, T^O) - \bar{\sigma}^*(t, T))d\bar{W}_t. \end{aligned}$$

Then, in view of (17), we have that, under $\mathbb{P}^F$,

$$dB^F(t, T) = \left(-\frac{1}{\bar{u}_{T^O}(t, r_t)}\frac{\partial \bar{u}_{T^O}}{\partial r}(t, r_t) + \frac{1}{\bar{u}_T(t, r_t)}\frac{\partial \bar{u}_T}{\partial r}(t, r_t)\right)\bar{\gamma}(t, r_t)B^F(t, T)d\bar{W}_t.$$

We recall that we suppose that $\bar{\gamma}(t, r_t) > 0$ for all t, so that the filtrations generated by $(\bar{W}_t)$ and (r_t) are identical. Therefore, as shown in Revuz and Yor (2005), a sufficient condition for $(B^F(t,T))$ having the predictable representation property is that

$$-\frac{1}{\bar{u}_{TO}(t,r_t)}\frac{\partial \bar{u}_{TO}}{\partial r}(t,r_t)+\frac{1}{\bar{u}_T(t,r_t)}\frac{\partial \bar{u}_T}{\partial r}(t,r_t)>0 \text{ a.s.}$$

□

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